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Global Optimality Properties of Total Annualized and Operating Cost Problems for Compressor Sequences

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The minimum total annualized cost problem for a series of nonisentropic compressors and coolers that brings a gas with constant compressibility factor from a specified initial pressure and temperature to a specified final pressure and the same temperature is studied in this work. It is established analytically that at the global optimum, the cooler outlet temperatures are equal to the minimum allowable temperature. For constant heat capacity, constant compressibility factor gases, additional properties of the globally optimal compressor sequence are analytically established for the minimum operating cost case. The aforementioned properties permit development of a solution strategy that identifies the globally minimum operating cost. Several case studies are presented to illustrate the developed theorems and solution strategies. © 2014 American Institute of Chemical Engineers *AIChE J*, 60: 4134–4149, 2014
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Introduction

According to the US Energy Information Administration (EIA),[1] between 2006 and 2010, the US shale gas production exhibited an average annual growth rate of 48%. In its 2013 Annual Energy Outlook for the 2010 to 2040 time period,[2] the EIA predicts a 113% increase in the production of shale gas, and an annual growth rate of 11.9% for natural gas consumption for transportation. This increased natural gas production, and the need for transportation of this natural gas across the country will place increased emphasis on gas compression systems. Combined with the increased use of compressed natural gas and compressed hydrogen for automotive transportation, and the extensive use of compression systems in the process industries, a compelling case arises for the optimization of compression systems.

Compressors contribute significantly to both the operating and capital cost of processing systems in which they are used. Since their operation often leads to a temperature increase of the gas being compressed, which in turn affects negatively compressor operation and increases power consumption, they are always operated in conjunction with a cooling system. The operating cost of each compressor is associated with its power consumption, while its capital cost is given by a power law expression of its power consumption.[3] The operating cost of the cooling system is proportional to the coolant flow rate, while its capital cost is much smaller than that of the compressor and is thus typically ignored. Given the large contribution of compressor energy consumption and operating costs to the overall energy consumption and operating cost of process plants, even small energy savings in compressor operations can have a significant beneficial impact. As an example, substitution of low efficiency with high efficiency compressors can reduce power consumption by over 5%.[4]

Minimization of total annualized cost (TAC) is a challenging problem with few global optimality results available in the literature. Martin and Manousiouthakis,[5] established rigorous optimality properties for the heat exchanger network TAC problem. Zhou and Manousiouthakis[6] provided converging upper and lower bounds to the minimum TAC problem for reactor networks within the IDEAS framework. Motivated by this problem formulation, Manousiouthakis et al.[7] developed a branch-and-bound-based method that can identify in a finite number of steps, the global minimum of a concave power law objective function over a system of linear constraints. Concave power law objective functions with rational exponents can be transformed to rationally constrained rational programs, which Manousiouthakis and Sourlas[8] demonstrated how to solve globally by first transforming them to convex, quadratically constrained quadratic programs with an additional separable concave constraint, and then solving using branch and bound[9] or generalized benders decomposition[10,11] methods.

The behavior of compressors and coolers is captured through models well-established in the literature.[12–15] Elrod,[16] Happel,[13] and Aris et al.[17] presented the solution to the steady state, work (power) minimization problem, for two, three, and a sequence, respectively, of isentropic compressors and intermediate coolers, that brings an ideal gas from an initial temperature and pressure to a desired final pressure and a final temperature equal to the initial temperature. The optimal works of the compressors are shown to be equal at the optimum. Wang and Fan[18] showed that the multistage, isentropic compression of ideal gas is part of a class of so-called one-dimensional (1-D) multistage processes, whose common characteristic is that they optimally require equal amounts of control action in each stage.

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In this work, the minimum TAC and minimum operating cost problems are formulated, for a sequence of compressors and coolers that brings a gas with constant compressibility factor from a specified initial state (T_0, P_0) to a specified final state (T_0, P_n). Using a constant compressibility factor other than unity is a common approach to account for gas nonidealities. In particular, using an average compressibility factor value between the values at (T_0, P_0), and (T_0, P_n) is common industrial practice.[19] To establish a range of variation for the compressibility factor of various gases, first methane is considered. The compressibility factor Z for methane varies between 0.70 and 1.05 for pressures between 0 psia (0 bar) and 3500 psia (241 bar), and temperatures between 32°F (0°C) and 400°F (204°C).[12] For hydrogen, which is the focus of our case study, the compressibility factor at the temperatures $T=300\text{K}, T=400\text{K}$ and for pressures such that $700\text{bar} \geq P \geq 1\text{bar}$ is such that $1.45 \geq Z \geq 1$ and $1.34 \geq Z \geq 1$, respectively.[20] For general gases, a popular compressibility factor correlation that exhibits errors of 2%–3% for nonpolar/slightly polar gases (though larger errors for polar/associative gases) is the Pitzer correlation $Z=Z^0+\omega Z^1$.[12] Values for the acentric factor ω typically range in the 0.1–0.7 range (methane 0.012, hydrogen −0.216).[12] Ranges for $Z^0(T_r, P_r), Z^1(T_r, P_r)$ can be determined from ranges for the reduced temperature and pressure T_r, P_r of the considered gas from the Lee/Kesler tables[12]

$$\left\{ \begin{array}{l} 4.00 \geq T_r \geq 1.15 \\ 1.00 \geq P_r \geq 0.01 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} 1.0115 \geq Z^0(T_r, P_r) \geq 0.7443 \\ 0.0864 \geq Z^1(T_r, P_r) \geq 0.0002 \end{array} \right\}.$$

In turn, this implies that within the above identified reduced temperature and pressure ranges, the compressibility factor for most gases is between 0.70 and 1.05.

The rest of the article is structured as follows: first, thermodynamic and economic models for the compression of a constant compressibility factor gas are developed. Second, it is established analytically that at the TAC problem's global minimum, the cooler outlet temperatures are equal to the minimum allowable temperature. For constant heat capacity, constant compressibility factor gases, additional properties of the globally optimal compressor sequence are analytically established for the minimum operating cost case. The aforementioned properties permit development of analytical formulas that enable the global solution of the minimum operating cost problem. Two case studies are presented to illustrate the developed theorems and solution strategies. Finally, conclusions are drawn.

Conceptual Framework

Preliminaries

In this work, the minimum cost problem is considered, for a sequence of compressors and isobaric coolers (shown in Figure 1) that brings a gas with constant compressibility factor Z from a specified initial state (T_0, P_0) to a specified final state (T_0, P_n). The inlet and outlet temperatures for any compressor in the series must be above T_0 and below $T_{\max}$, the maximum allowable operating temperature for all compressors, respectively. The following are considered to hold:

1. The compressors operate with isentropic efficiency $\eta \in (0,1)$ (i.e., not isentropically)
2. The coolant inlet and outlet temperatures are considered fixed and known for all coolers. They are defined such that $T_{c,i,\text{in}} \leq T_{c,i,\text{out}} \leq T_0, i=1,n$.

Thermodynamic relations

Lemma.

a. The changes in molar enthalpy and molar entropy of a real fluid from the state (T^o, P^o) to the state (T, P) are

$$H(T,P)-H(T^o,P^o)=\left[\begin{array}{c} \int_{P^R}^{P} V\left(T,P'\right)\left(1-\beta\left(T,P'\right)T\right)dP' + \int_{T^o}^{T} C_p\left(T',P^R\right)dT' + \\ + \int_{P^o}^{P^R} V\left(T^o,P'\right)\left(1-\beta\left(T^o,P'\right)T^o\right)dP' \end{array}\right] \quad (1)$$

$$S(T,P)-S(T^o,P^o)=\left[\begin{array}{c} -\int_{P^R}^{P} \beta\left(T,P'\right)V\left(T,P'\right)dP' + \int_{T^o}^{T} \frac{C_p\left(T',P^R\right)}{T'}dT' + \\ -\int_{P^o}^{P^R} \beta\left(T^o,P'\right)V\left(T^o,P'\right)dP' \end{array}\right] \quad (2)$$

where (T^R, P^R) denotes a reference state where $P^R \to 0$ (ideal gas state), $\beta(T,P) \triangleq \frac{1}{V(T,P)}\frac{\partial V(T,P)}{\partial T}$, $\beta(T,P) \triangleq \frac{1}{V(T,P)}\frac{\partial V(T,P)}{\partial T}$, $C_p(T,P^R) \triangleq C_p(T) \geq 0 \forall T \in \Re^+$ is the molar, constant pressure $P^R \to 0$, ideal gas, heat capacity of a fluid, that is only a function of temperature.

b. A gas featuring a constant compressibility factor satisfies the following

$$Z(T,P) \triangleq \frac{PV(T,P)}{RT} = Z = \text{constant} \quad (3)$$

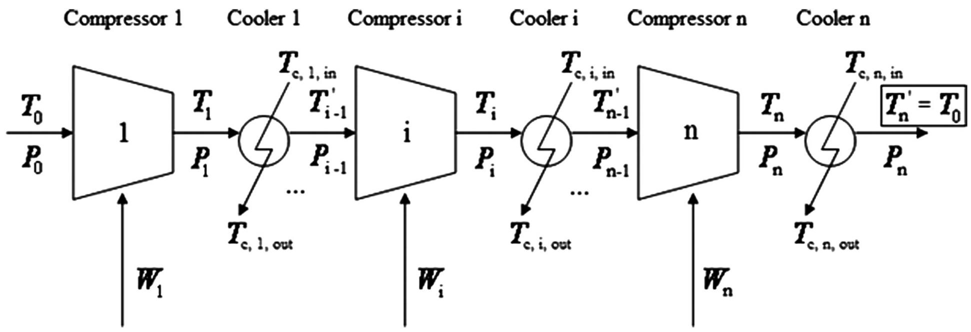

Figure 1. Process flowsheet for *n* compressors and *n* intermediate coolers.

$$\beta(T,P)=\beta(T)=\frac{1}{T},\kappa(T,P)=\kappa(P)=\frac{1}{P} \tag{4}$$

$$H(T,P)-H(T^o,P^o)=\int_{T^o}^{T} C_p\left(T',P^R\right)dT' \tag{5}$$

$$S(T,P)-S(T^o,P^o)=\int_{T^o}^{T}\frac{C_p\left(T',P^R\right)}{T'}dT'-ZR\ln\left(\frac{P}{P^o}\right) \tag{6}$$

$$C_p(T,P)-C_v(T,P)=RZ \tag{7}$$

c. Consider a constant compressibility factor gas compressed through a reversible adiabatic (ideal) compressor, with inlet and outlet temperatures and pressures T_{in},T'_{out} and P_{in},P_{out}, respectively. The compressor consumes the following amount of molar work, and satisfies the following isentropic requirement across its inlet and outlet

$$W_{id}=R\frac{H\left(T'_{out},P_{out}\right)-H(T_{in},P_{in})}{R}=R\int_{T_{in}}^{T'_{out}}\frac{C_p\left(T'\right)}{R}dT' \tag{8}$$

$$S\left(T'_{out},P_{out}\right)=S(T_{in},P_{in})\iff\int_{T_{in}}^{T'_{out}}\frac{C_p\left(T'\right)}{RT'}dT'=Z\ln\left(\frac{P_{out}}{P_{in}}\right) \tag{9}$$

d. Consider a constant compressibility factor gas compressed through an adiabatic (real) compressor with known efficiency $\eta\in(0,1)$, and inlet and outlet temperatures and pressures T_{in},T_{out} and P_{in},P_{out}, respectively. The compressor consumes the following amount of molar work, and satisfies the following efficiency relations with the ideal compressor

$$W_r=R\frac{H(T_{out},P_{out})-H(T_{in},P_{in})}{R}=R\int_{T_{in}}^{T_{out}}\frac{C_p\left(T'\right)}{R}dT' \tag{10}$$

$$\eta_i=\frac{W_{id}}{W_r}=\frac{H\left(T'_{out},P_{out}\right)-H(T_{in},P_{in})}{H(T_{out},P_{out})-H(T_{in},P_{in})}=\frac{\int_{T_{in}}^{T'_{out}}C_p\left(T'\right)dT'}{\int_{T_{in}}^{T_{out}}C_p\left(T'\right)dT'} \tag{11}$$

e. The changes in molar enthalpy and molar entropy of a constant compressibility factor gas with a temperature-independent (constant), constant-pressure, ideal gas, heat capacity $C_p(T,P^R)\triangleq C_p(T)=C_p=$constant, from the state (T^o,P^o) to the state (T,P) are

$$\frac{H(T,P)-H(T^o,P^o)}{R}=\frac{C_p}{R}(T-T^o) \tag{12}$$

$$\frac{S(T,P)-S^o(T^o,P^o)}{R}=\frac{C_p}{R}\ln\left(\frac{T}{T^o}\right)-Z\ln\left(\frac{P}{P^o}\right) \tag{13}$$

f. Let this gas be compressed through an adiabatic ideal compressor, and through an adiabatic real compressor with known efficiency $\eta\in(0,1)$. Let the inlet temperatures, and inlet and outlet pressures T_{in},P_{in},P_{out} to both compressors be the same, and let the outlet temperatures be denoted as T'_{out},T_{out}, respectively. Then the following relations hold

$$\begin{aligned}W_{id}&=ZR\frac{k}{k-1}T_{in}\left(\left(\frac{P_{out}}{P_{in}}\right)^{\frac{k-1}{k}}-1\right),\\W_r&=\frac{1}{\eta}ZR\frac{k}{k-1}T_{in}\left(\left(\frac{P_{out}}{P_{in}}\right)^{\frac{k-1}{k}}-1\right)\end{aligned} \tag{14}$$

$$T'_{out}=T_{in}\left(\frac{P_{out}}{P_{in}}\right)^{\frac{k-1}{k}},T_{out}=T_{in}\left(1+\frac{\left(\frac{P_{out}}{P_{in}}\right)^{\frac{k-1}{k}}-1}{\eta}\right) \tag{15}$$

Proof. See Appendix.

The above thermodynamic properties are used in establishing a monotonicity property regarding the behavior of a real compressor.

Theorem 1. *Consider the compression of a gas with constant compressibility factor $Z>0$, by an ideal compressor and by a real compressor with known efficiency η. Let the gas inlet temperature T_{in}, the gas inlet pressure P_{in}, and the compression ratio $\frac{P_{out}}{P_{in}}>1$ be the same for both compressors. Finally, let the outlet temperatures of the ideal compressor and the real compressor be denoted as T'_{out} and T_{out}, respectively. Then:*

1. Let the compression ratio $\frac{P_{out}}{P_{in}}>1$ be known, and $\int_{T_{in}}^{T'_{out}}\frac{C_p(T')}{T'}dT'$ be equal to the positive constant $ZR\ln\left(\frac{P_{out}}{P_{in}}\right)>0$. Then, there exists a function $f:\Re^+\to\Re^+$, $f:T_{in}\to T'_{out}=f(T_{in})$. In addition, the function f is differentiable and monotonically increasing with derivative

$$\frac{df(T_{in})}{dT_{in}}=\frac{C_p(T_{in})}{T_{in}}\frac{f(T_{in})}{C_p(f(T_{in}))}=\frac{C_p(T_{in})}{T_{in}}\frac{T'_{out}}{C_p\left(T'_{out}\right)}>0\,\forall T_{in}>0. \tag{16}$$

2. Let the compression ratio $\frac{P_{out}}{P_{in}}>1$ be known, and $\int_{T_{in}}^{T'_{out}}\frac{C_p(T')}{T'}dT'$ be equal to the positive constant $ZR\ln\left(\frac{P_{out}}{P_{in}}\right)>0$. Then the function

$$\Delta H:\Re^+\rightarrow\Re^+,\Delta H:T_{\text{in}}\rightarrow\Delta H(T_{\text{in}})\triangleq H\left(T'_{\text{out}},P_{\text{out}}\right)$$
$$-H(T_{\text{in}},P_{\text{in}})=H(f(T_{\text{in}}),P_{\text{out}})-H(T_{\text{in}},P_{\text{in}})=\int_{T_{\text{in}}}^{T'_{\text{out}}}C_{\text{p}}\left(T'\right)dT'$$

is differentiable and monotonically increasing with derivative

$$\begin{aligned}\frac{d(\Delta H(T_{\text{in}}))}{dT_{\text{in}}}&=C_{\text{p}}(T_{\text{in}})\left(\frac{f(T_{\text{in}})}{T_{\text{in}}}-1\right)\\&=C_p(T_{\text{in}})\left(\frac{T'_{\text{out}}}{T_{\text{in}}}-1\right)>0\,\forall T_{\text{in}}>0.\end{aligned}\tag{17}$$

Proof. See Appendix.

Mathematical problem formulation

The optimization problem considered, in this work, is the minimization of an objective function that reflects the TAC of the compressor/cooler sequence, subject to a number of constraints that capture the behavior of the sequence units, and the operating requirements on these units. The general mathematical formulation of the problem is

$$\begin{aligned}&v=\min_{\substack{\{W_i\}_{i=1}^n\\\{\dot n_{c,i}\}_{i=1}^n\\\{P_i,T_i,T'_i,T''_i\}_{i=1}^n}}\sum_{i=1}^{n}\left[FC^{\text{cap.}}_{\text{compr.}}(W_i\cdot\dot n)^a+C^{\text{oper.}}_{\text{compr.}}(W_i\cdot\dot n)+C^{\text{oper.}}_{\text{cooler}}(\dot n_{\text{c},i})\right]\\&\text{s.t.}\\&W_i=R\int_{T'_{i-1}}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'=\frac{R}{\eta_i}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT',i=1,n\\&Z\ln\left(\frac{P_i}{P_{i-1}}\right)=\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{RT'}dT',i=1,n;\frac{P_n}{P_0}\text{ known}\\&\dot n_{\text{c},i}=\frac{\dot n\cdot R\int_{T'_i}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{\text{c,out}}-T_{\text{c,in}}\right)},i=1,n\\&\eta_i=\frac{\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'}{\int_{T'_{i-1}}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'},i=1,n\\&T'_{i-1}\le T''_i\le T_i\le T_{\max}<\infty,i=1,n;T'_0=T_0\\&0<T_0\le T'_i\le T_i,i=1,n-1;T'_n=T_0\end{aligned}\tag{18}$$

The objective function is a finite sum of terms that reflect the TAC associated with the compressor/cooler units. Each term of this sum consists of three components: the *i*th compressor's annualized capital cost, the *i*th compressor's annual operating cost, and the *i*th cooler's annual operating cost. The capital cost of coolers is considered to be small compared to that of compressors and is thus not incorporated in the problem formulation. The capital cost of each compressor is considered to be given by a power law expression of its power consumption. The operating cost of the compressor is considered proportional to its power consumption, while the operating cost of the cooler is proportional to its coolant flow rate.

The first equality constraint quantifies the work consumed by the *i*th real compressor, in terms of the inlet and outlet temperatures of the *i*th real and *i*th ideal (isentropic) compressor, respectively. The second equality constraint is derived based on the isentropic requirement for an ideal compressor, and quantifies the relationship between the inlet and outlet pressures and temperatures of an ideal (isentropic) compressor. It also states the requirement that the compressor sequence's overall compression ratio $\frac{P_n}{P_0}$ is known. The third equality constraint is based on the first law of thermodynamics for the *i*th cooler, equating the coolant and compressed gas heat loads in the *i*th cooler. The fourth equality constraint relates the efficiency of the *i*th real compressor to the molar enthalpy changes across the *i*th ideal and real compressors.

The first set of inequalities stipulates that the *i*th (ideal or real) compressor's inlet temperature must be below the *i*th ideal compressor's outlet temperature, which must be below the *i*th real compressor's outlet temperature, which must be below the maximum allowable temperature. The second set of inequalities imposes the restriction that the outlet gas temperature of the *i*th cooler should be below the *i*th real compressor's outlet temperature and above the compressor sequence's inlet and outlet temperature T_0.

The first and third sets of equality constraints can be used to substitute for W_i and $\dot n_{\text{c},i}$ in the objective function. In addition, the second set of equality constraints can be substituted by a single equality constraint involving only the known overall compression ratio $\frac{P_n}{P_0}$. Then, the above optimization problem (18) becomes

$$\begin{aligned}&v=\min_{\left\{T_i,T'_i,T''_i\right\}_{i=1}^n}\sum_{i=1}^{n}\left[\begin{array}{l}FC^{\text{cap.}}_{\text{compr.}}\left(\frac{\dot n\cdot R}{\eta_i}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'\right)^a+C^{\text{oper.}}_{\text{compr.}}\left(\frac{\dot n\cdot R}{\eta_i}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'\right)+\\+C^{\text{oper.}}_{\text{cooler}}\left(\frac{\dot n\cdot R\int_{T'_i}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{c,\text{out}}-T_{c,\text{in}}\right)}\right)\end{array}\right]\\&\text{s.t.}\\&Z\ln\left(\frac{P_n}{P_0}\right)=\sum_{i=1}^{n}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{RT'}dT'\\&0<\eta_i=\frac{\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'}{\int_{T'_{i-1}}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'}<1,i=1,n\\&T'_{i-1}\le T''_i\le T_i\le T_{\max}<\infty,i=1,n;T'_0=T_0\\&0<T_0\le T'_i\le T_i,i=1,n-1;T'_n=T_0\end{aligned}\tag{19}$$

Next, it is shown that the optimization problem (19) possesses the following optimality property, allowing for significant reduction of dimensionality of the problem:

Theorem 2. *$T'_{i-1}=T_0$ for $i=2,n$ at the global optimum of* (19).

Proof. See Appendix.

In light of Theorem 2, we can now replace all T'_{i-1} terms in our problem with T_0. Our resulting problem is

$$v=\min_{\{T_i,T''_i\}_{i=1}^n}\sum_{i=1}^{n}\left[\begin{array}{l} FC_{\text{compr.}}^{\text{cap.}}\left(\frac{\dot{n}\cdot R}{\eta_i}\int_{T_0}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'\right)^a+C_{\text{compr.}}^{\text{oper.}}\left(\frac{\dot{n}\cdot R}{\eta_i}\int_{T_0}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'\right)+\\ +C_{\text{cooler}}^{\text{oper.}}\left(\frac{\dot{n}\cdot R\int_{T_0}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}(T_{c,\text{out}}-T_{c,\text{in}})}\right)\end{array}\right]$$
$$\text{s.t.}$$
$$Z\ln\left(\frac{P_n}{P_0}\right)=\sum_{i=1}^{n}\int_{T_0}^{T''_i}\frac{C_{\text{p}}(T')}{RT'}dT' \tag{20}$$
$$\eta_i=\frac{\int_{T_0}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'}{\int_{T_0}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'},\ i=1,n$$
$$0<T_0\le T''_i\le T_i\le T_{\max}<\infty,\ i=1,n$$

Constant heat capacity formulation

Compressor sequences use intercoolers so that compressor exit temperatures are not allowed to rise significantly. This suggests as a reasonable approximation, the use of a temperature-independent, constant pressure, ideal gas heat capacity with values equal to the average value of the temperature-dependent, constant pressure, ideal gas heat capacity over the temperature interval of the minimum and maximum allowable compressor outlet temperatures.

When C_{p} is constant (or equivalently $k\triangleq\frac{C_{\text{p}}}{C_{\text{v}}}=$constant), problem (20) becomes

$$v=\left\{\begin{array}{l}\min_{\{w_i\}_{i=1}^n} A\sum_{i=1}^{n}\frac{1}{\eta_i}\cdot w_i+B\sum_{i=1}^{n}\left(\frac{1}{\eta_i}\right)^a\cdot(w_i)^a\\ \text{s.t.}\\ \prod_{i=1}^{n}(w_i+1)-C=0\\ 0\le w_i\le\eta_iD,\ i=1,n\end{array}\right\} \tag{21}$$

where, $A\triangleq\left(C_{\text{compr.}}^{\text{oper.}}\cdot\dot{n}\cdot C_{\text{p}}+\frac{C_{\text{cooler}}^{\text{oper.}}\cdot\dot{n}\cdot C_{\text{p}}}{C_{\text{p,c}}(T_{\text{c,out}}-T_{\text{c,in}})}\right)T_0\ge 0$, $B\triangleq F C_{\text{compr.}}^{\text{cap.}}\dot{n}^a\cdot(C_{\text{p}})^a(T_0)^a\ge 0$, $C\triangleq\left(\frac{P_n}{P_0}\right)^{\left(\frac{ZR}{C_{\text{p}}}\right)}>1$, $D\triangleq\frac{T_{\max}-T_0}{T_0}>0$, and $w_i\triangleq\frac{T''_i-T_0}{T_0}\iff T''_i=w_i\cdot T_0+T_0=T_0\cdot(w_i+1)$.

Theorem 3. *Let $D>0, C>1, \eta_i\in(0,1]\ \forall i=1,n$. Then the optimization problem* (21) *is feasible iff $C\le\prod_{i=1}^{n}(\eta_iD+1)$.*

Proof. See Appendix

Operating costs only

Define the sets $S_D^w\triangleq\{i=1,n:w_i=\eta_iD\}$ (compressors operating at maximum allowable outlet temperature), $S_0^w\triangleq\{i=1,n:w_i=0\}$ (compressors not in use), and $S_I^w\triangleq\{\{1,\cdots,n\}-S_D^w-S_0^w\}$ (compressors in use and operating below maximum allowable outlet temperature) with cardinalities N_D^w,N_0^w,N_I^w, respectively. Then $N_I^w\triangleq n-N_D^w-N_0^w$. If $N_I^w=0$, it is clear that $N_D^w\ge 1$, otherwise the compression level C could not be attained. In this case, straightforward combinatorial calculations, on which compressors belong to S_D^w, can be carried out to identify the global minimum without any need of the optimality conditions. Thus in the Theorem below, it is considered that $N_I^w\ge 1$.

Theorem 4. *Let $A>0, B=0, D>0, 1<C\le\prod_{i=1}^{n}(\eta_iD+1)$, $\eta_i\in(0,1]\forall i=1,n,N_I^w\ge 1$ and consider the problem*

$$v=A\cdot\left\{\begin{array}{l}\min_{\{w_i\}_{i=1}^n}\sum_{i=1}^{n}\frac{1}{\eta_i}\cdot w_i\\ \text{s.t.}\\ \prod_{i=1}^{n}(w_i+1)-C=0\\ 0\le w_i\le\eta_iD,\ i=1,n\end{array}\right\}.$$

Then, the optimum objective function value is

$$v=A\cdot\left[N_I^w\frac{C^{\frac{1}{N_I^w}}}{\left(\prod_{l\in S_I^w}\eta_l\right)^{\frac{1}{N_I^w}}\cdot\left(\prod_{m\in S_D^w}(\eta_mD+1)\right)^{\frac{1}{N_I^w}}}-\sum_{k\in S_I^w}\frac{1}{\eta_k}+N_D^w\cdot D\right] \tag{22}$$

the optimum variable values are

$$w_k=\left[\eta_k\left[\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\right]^{\frac{1}{N_I^w}}-1\right]\in(0,\eta_k D)\;\forall k\in S_I^w$$

$$w_i=0\;\forall i\in S_0^w$$

$$w_j=\eta_j D\;\forall j\in S_D^w$$

and the following four conditions must be satisfied by the global minimum

$$\max\left(\left(D+\frac{1}{\min_{j\in S_D^w}\eta_j}\right)^{N_I^w},\left(\frac{1}{\min_{k\in S_I^w}\eta_k}\right)^{N_I^w}\right)\le\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\cdot\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\le$$
$$\le\min\left(\left(\frac{1}{\max_{i\in S_0^w}\eta_i}\right)^{N_I^w},\left(D+\frac{1}{\max_{k\in S_I^w}\eta_k}\right)^{N_I^w}\right)$$
$$\text{if } N_0^w\neq0\wedge N_D^w\neq0$$

$$\max\left(\left(D+\frac{1}{\min_{j\in S_D^w}\eta_j}\right)^{N_I^w},\left(\frac{1}{\min_{k\in S_I^w}\eta_k}\right)^{N_I^w}\right)\le\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\cdot\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\le$$
$$\le\left(D+\frac{1}{\max_{k\in S_I^w}\eta_k}\right)^{N_I^w}\;\text{if } N_0^w=0\wedge N_D^w\neq0$$

$$\left(\frac{1}{\min_{k\in S_I^w}\eta_k}\right)^{N_I^w}\le\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)}$$
$$\le\min\left(\left(\frac{1}{\max_{i\in S_0^w}\eta_i}\right)^{N_I^w},\left(D+\frac{1}{\max_{k\in S_I^w}\eta_k}\right)^{N_I^w}\right)\text{if } N_0^w\neq0\wedge N_D^w=0$$

$$\left(\frac{1}{\min_{k\in S_I^w}\eta_k}\right)^{n}\le\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)}\le\left(D+\frac{1}{\max_{k\in S_I^w}\eta_k}\right)^{n}\tag{23}$$
$$\text{if } N_0^w=0\wedge N_D^w=0$$

Proof. See Appendix.

Theorem 4 suggests that at the global optimum all interior compressors must be such that the ratio of the exit temperature of the corresponding isentropic compressor over the efficiency of the real compressor is the same for all compressors. This conclusion arises from the fact that $\frac{w_k+1}{\eta_k}=\frac{T''_k}{T_0\eta_k}=\left[\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\right]^{\frac{1}{N_I^w}}\;\forall k\in S_I^w$. Furthermore, Theorem 4 also suggests that the following compressor efficiency-related properties must hold for the three defined sets S_D^w, S_0^w, S_I^w

$$\max\left(\left(D+\frac{1}{\min_{j\in S_D^w}\eta_j}\right)^{N_I^w},\left(\frac{1}{\min_{k\in S_I^w}\eta_k}\right)^{N_I^w}\right)$$
$$\le\frac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\cdot\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}$$
$$\le\min\left(\left(\frac{1}{\max_{i\in S_0^w}\eta_i}\right)^{N_I^w},\left(D+\frac{1}{\max_{k\in S_I^w}\eta_k}\right)^{N_I^w}\right)$$
$$\text{if } N_0^w\neq0\wedge N_D^w\neq0$$

1. The maximum compressor efficiency in the set of unused compressors has to be less than or equal to the minimum compressor efficiency in the set of compressors used below capacity.
2. The sum of the temperature defined bound D and the inverse of the maximum compressor efficiency in the set of compressors used below capacity has to be greater than or equal to the inverse of the minimum compressor efficiency in the set of compressors used below capacity.
3. The inverse of the maximum compressor efficiency in the set of unused compressors has to be greater than or equal to the sum of the temperature defined bound D and the inverse of the minimum compressor efficiency in the set of compressors used at capacity.
4. The maximum compressor efficiency in the set of compressors used below capacity has to be less than or equal to the minimum compressor efficiency in the set of compressors used at capacity.

When a set's cardinality is zero, the above criteria involving that set should be ignored. In particular, $N_0^w=0$ implies that criteria 1, 3 should be ignored; $N_I^w=0$ implies that criteria 1, 2, 4 should be ignored; and finally $N_D^w=0$ implies that criteria 3, 4 should be ignored.

The above suggest the following procedure to identify the global minimum:

1. Rank from lowest to highest the inverses of the compressor efficiencies.
2. Select a combination of cardinalities N_D^w, N_I^w for the sets S_D^w, S_I^w respectively, possibly starting from zero and such that $N_D^w+N_I^w\le n$. If all combinations of cardinalities have been considered, then go to step 5.
3. Consider that the N_D^w compressors with the highest efficiencies belong to S_D^w, the N_I^w compressors with the next highest efficiencies belong to S_I^w, and the remaining $N_0^w=n-N_D^w-N_I^w$ compressors belong to S_0^w

4. Verify that the four aforementioned compressor efficiency-related properties of S_D^w, S_0^w, S_I^w (which are independent of the value of C) are satisfied. If not, then declare the combination infeasible, go to step 2 and consider another cardinality N_D^w, N_I^w combination. If yes, then store this combination in a feasible candidate combination list.

5. For a given value of C, and for each combination of cardinalities N_D^w, N_I^w in the feasible candidate combination list, verify that the necessary conditions of optimality (23) are satisfied. If no, then go to the next combination of cardinalities N_D^w, N_I^w in the feasible candidate combination list and repeat. If yes, then evaluate v using Eq. 22 and store it in a candidate optimum list. Then go to the next combination of cardinalities N_D^w, N_I^w in the feasible candidate combination list and repeat until the list is exhausted.

6. Select the minimum value of v from the candidate optimum list. This is the global minimum v.

Discussion

The above global optimum solution procedure requires that a number of cases be considered depending on the three cardinalities N_0^w, N_D^w, N_I^w, which must also satisfy $N_0^w + N_D^w + N_I^w = n$. Thus the number of cases that must be considered grows at most quadratically with the number of compressors n. For each of these cases, the efficiency-related optimality criteria of Theorem 4 significantly reduce the number of alternatives that need to be considered. The above facts make the solution procedure effective, even for large numbers of compressors.

Theorem 4 and the solution procedure discussed in the previous section simplify greatly for the case of compressors of equal efficiency, that is, for the case $\eta_i = \eta \forall i = 1, n$. Then Theorem 4 implies that the optimum objective function value is

$$v = A \cdot \left[\frac{N_I^w}{\eta} \left(\left(\frac{C}{(\eta D + 1)^{N_D^w}} \right)^{\frac{1}{N_I^w}} - 1 \right) + N_D^w \cdot D \right],$$

the optimal variable values are

$$w_k = \left[\left[\frac{C}{(\eta D + 1)^{N_D^w}} \right]^{\frac{1}{N_I^w}} - 1 \right] \in (0, \eta D) \, \forall k \in S_I^w$$

$$w_i = 0 \, \forall i \in S_0^w$$

$$w_j = \eta_j D \, \forall j \in S_D^w$$

and the necessary optimality conditions for $N_I^w \geq 1$ become

$$\left(D + \frac{1}{\eta} \right)^{N_I^w} \leq \frac{C}{\eta^{N_I^w} (\eta D + 1)^{N_D^w}} \leq \left(\frac{1}{\eta} \right)^{N_I^w} \text{ if } N_0^w \neq 0 \wedge N_D^w \neq 0$$

$$\left(D + \frac{1}{\eta} \right)^{N_I^w} = \frac{C}{\eta^{N_I^w} (\eta D + 1)^{N_D^w}} \text{ if } N_0^w = 0 \wedge N_D^w \neq 0$$

$$\left(\frac{1}{\eta} \right)^{N_I^w} = \frac{C}{\eta^{N_I^w}} \text{ if } N_0^w \neq 0 \wedge N_D^w = 0$$

$$\left(\frac{1}{\eta} \right)^{n} \leq \frac{C}{\eta^{n}} \leq \left(D + \frac{1}{\eta} \right)^{n} \text{ if } N_0^w = 0 \wedge N_D^w = 0$$

It becomes obvious from the above that for the case of equal efficiency compressors, and when $N_I^w \geq 1$, then the case $N_0^w \neq 0 \wedge N_D^w \neq 0$ is impossible. Also that, at the optimum, all used compressors whose outlet temperature is below the maximum operating temperature must have equal power consumption and equal exit temperature.

Table 1. Parameters for Both Case Studies

Parameter (Units)	Value	Parameter (Units)	Value
T_0(K)	298	P_0(kPa)	101.325
$T_{\max}$ (K)	405	C_p(J/mol · K)	28.85
$D = \frac{T_{\max} - T_0}{T_0}$	0.3591	Z	1

Case Studies

We now present two case studies involving compression of a gas with compressibility factor equal to one, from the initial state $(T_0, P_0) = (298\text{K}, 101.325\text{kPa})$ to the final state $(T_n, P_n) = (T_0, P_n) = (298\text{K}, P_n)$, where P_n varies. In both cases, the parameters shown in Table 1 are fixed.

The first case study considers compressors of equal efficiency, and examines how the globally minimum operating cost value changes with the number of available compressors and with varying final pressures P_n. In this case, the total number of available compressors to be studied will be one, two, and three.

The second case study considers compressors of unequal efficiencies, and examines how the globally minimum operating cost value changes with the final pressure P_n. In this case, the total number of available compressors to be studied is always four. However, the optimal sequence may not necessarily use all of them.

Both case studies are solved using the solution procedure suggested following Theorem 4 in the previous section.

Case Study 1: Operating cost minimization, compressors with equal efficiencies

For this case study, we consider compressors with the same efficiency $\eta = 1$. Figures 2 and 3 illustrate the objective function values for one, two, and three compressors of equal efficiencies in series. Figure 3 is a magnification of region 1 to emphasize the differences in the objective function values for each of the compressor systems considered. For all desired outlet pressures in which one compressor is feasible (region 1), and those in which two compressors are feasible (regions 1 and 2), the objective value corresponding to the three equal compressors is the lowest. This is in agreement with Theorem 4, which suggests that when $N_I^w \geq 1$, then all compressors operating below the maximum operating temperature must be equal, and that it is impossible to have $N_0^w \neq 0 \wedge N_D^w \neq 0$.

Case Study 2: Operating cost minimization, compressors with unequal efficiencies

For this case, we considered a system using four compressors of unequal efficiencies ($\eta_1 = 1, \eta_2 = 0.9, \eta_3 = 0.8$, and $\eta_4 = 0.7$) to explore how the global optimum prioritizes their use. Figure 4 illustrates the globally optimal objective

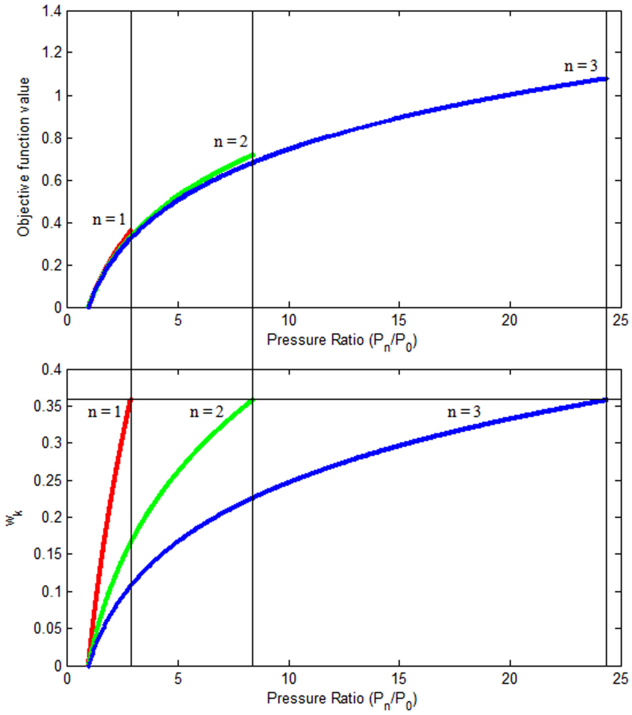

Figure 2. Optimum objective function and variable values for various *n*.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

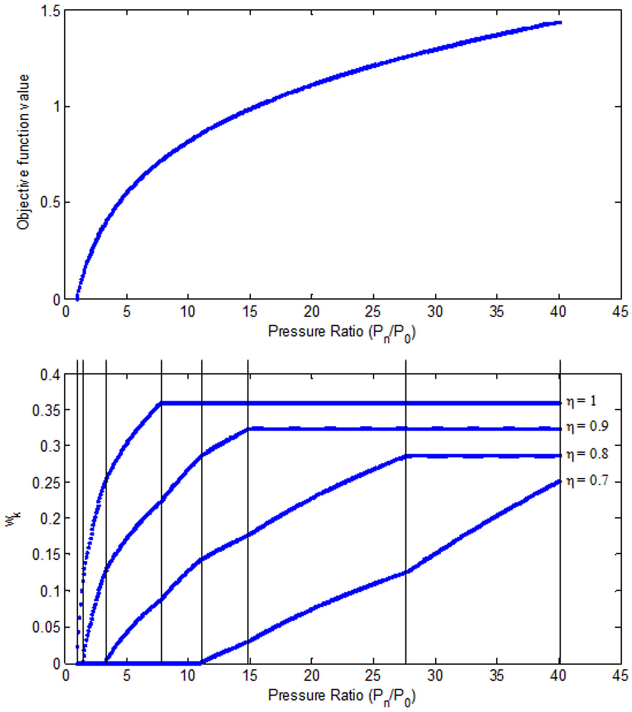

Figure 4. Global optima for Example 2 (unequal compressors).

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

function values and temperature ratios $\{w_i\}_{i=1}^4$ for this system identified via the solution method from Theorem 4. Vertical lines identify pressure ratios at which either one compressor begins to be used, or when one compressor reaches capacity; that is, when the cardinalities N_D^w, N_0^w, N_I^w of the global optimum change. Table 2 depicts how much energy savings is achieved by switching from a system using all four compressors with equal duties to the identified globally optimal configuration.

From Figure 4, we can see that this collection of compressors in series can deliver compressed gas at pressures up to 40 times the initial pressure P_0. As the pressure ratio increases, the global optimum begins using the compressor with the highest efficiency ($\eta_1 = 1$) first, with the other compressors following in descending order of efficiency.

For a given pressure ratio, both the temperature ratios and slopes of the temperature ratio curves decrease with decreasing efficiency. These results are in line with intuition: given a set of compressors of varying efficiencies, it is best to allocate the bulk of the necessary work to the most efficient compressor, saving the other compressors for higher work demands. However, the number of compressors to be used at each pressure ratio is not straightforward to identify. It is also important to emphasize that, unlike the equal efficiency case, there are pressure ratio ranges for which it is not optimal to use all available compressors. As can be seen in Figure 4, below a pressure ratio of about 10, it is not optimum to use all four available compressors. In fact, below a value of 3.3, it is optimal to use only two compressors, and below a value of 1.4, it is optimal to use only one compressor. This behavior is completely different from the equal efficiency compressor case, where for all pressure ratios it is optimum to use all available compressors.

It is also important to emphasize that the energy savings that stem from using the optimal compressor sequence over

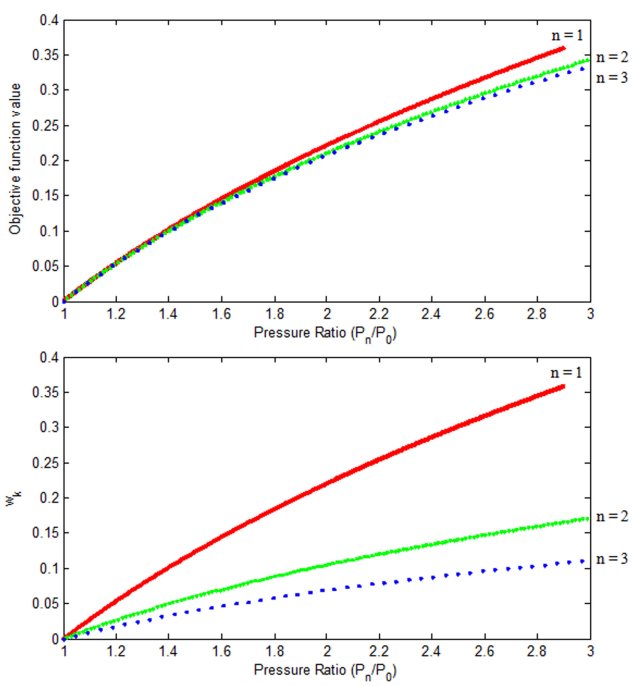

Figure 3. Magnification of optimum objective function and variable values for various *n*.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Table 2. Energy Savings from Use of Optimal Compressors over Equal Outlet Temperature Compressors

P_n/P_0	$W_{tot,eq}$(J/mol)	$W_{tot,opt}$(J/mol)	$W_{savings}$(%)
1.00	0	0	N/A
1.25	655	571	12.868
1.50	1198	1065	11.105
2.00	2064	1879	8.988
3.00	3334	3084	7.513
4.00	4252	3979	6.416
5.00	4979	4691	5.770
6.00	5580	5285	5.295
7.00	6095	5795	4.928
8.00	6545	6243	4.626
10.00	7307	7009	4.077
12.00	7943	7653	3.651
14.00	8485	8207	3.273
16.00	8959	8694	2.956
18.00	9382	9132	2.662
20.00	9758	9529	2.341
22.00	10,107	9894	2.110
24.00	10,428	10,232	1.881
26.00	10,723	10,546	1.654
28.00	10,997	10,840	1.430
30.00	11,253	11,118	1.194
32.00	11,497	11,384	0.984
34.00	11,725	11,638	0.745
36.00	11,941	11,881	0.501
38.00	12,146	12,115	0.254
40.00	12,341	12,341	0.006

a conventional design, such as a compressor sequence where all compressor outlet temperatures are equal, are not insignificant, especially for small pressure ratios for a given number of compressors. Table 2 below summarizes these savings, which can be as high as 12.868%.

Conclusions

In this work, we studied both the TAC and the minimum operating cost problems for a system of compressors and coolers in series bringing a gas with constant compressibility factor from a specified initial state (T_0, P_0) to a specified final state (T_0, P_n). We established analytically that at the global optimum of the general TAC problem, the cooler outlet temperatures are equal to the minimum allowable temperature. For constant heat capacity, constant compressibility factor gases, additional properties of the globally optimal compressor sequence are analytically established for the minimum operating cost case. The aforementioned properties permitted development of an analytical solution methodology that can identify the globally minimum operating cost for any number of compressors of possibly different efficiencies. Two case studies are presented to illustrate the developed theorems and solution strategies. It is shown that the globally minimum cost for sequences of compressors with unequal efficiencies may correspond to a sequence that does not use all available compressors. Energy savings of up to 12.868% are identified over conventional designs.

Acknowledgments

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Notation

Greek letters

β = volume expansivity (1/K)
η_i = efficiency of compressor i
κ = isothermal compressibility (1/kPa)
v = objective function value

Letters

A = operating cost coefficient; $A \triangleq \left(C^{oper.}_{compr.} \cdot \dot{n} \cdot C_p + \frac{C^{oper.}_{cooler} \cdot \dot{n} \cdot C_p}{C_{p,c}\left(T_{c,out} - T_{c,in}\right)}\right) T_0(\$)$
B = capital cost coefficient; $B \triangleq F C^{cap.}_{compr.} \dot{n}^a \cdot \left(C_p\right)^a (T_0)^a (\$)$
C = modified pressure ratio; $C \triangleq \left(\frac{P_n}{P_0}\right)^{\left(\frac{ZR}{C_p}\right)}$
C_p = constant-pressure molar heat capacity of gas (J/mol·K)
$C_{p,c}$ = constant-pressure molar heat capacity of coolant (J/mol·K)
C_v = constant-volume molar heat capacity of gas (J/mol·K)
$C^{cap.}_{compr.}$ = capital cost coefficient of compression ($\$/(W)^a$)
$C^{oper.}_{compr.}$ = operating cost coefficient of compression ($/J)
$C^{oper.}_{cooler}$ = operating cost coefficient of cooling ($/mol)
D = maximum normalized compressor outlet temperature; $D \triangleq \frac{T_{max} - T_0}{T_0}$
F = annualization factor (1/s)
H = molar enthalpy of fluid stream (J)
$\dot{n}$ = molar flow rate of gas stream (mol/s)
$\dot{n}_{c,i}$ = molar flow rate of coolant stream through cooler i (mol/s)
P_0 = inlet pressure of gas stream to compressor/cooler system (kPa)
P_i = outlet pressure of gas stream from compressor i (kPa)
P_n = outlet pressure of gas stream from compressor/cooler system (kPa)
R = universal gas constant (J/mol·K)
S = molar entropy of fluid stream (J/K)
T_0 = inlet temperature of gas stream to compressor/cooler system (K)
$T_{c,i,in}$ = inlet temperature of coolant to cooler i (K)
$T_{c,i,out}$ = outlet temperature of coolant from cooler i (K)
T_i = outlet temperature of gas stream from compressor i (K)
T'_{i-1} = outlet temperature of gas stream from cooler $i-1$ to compressor i (K)
T''_i = outlet temperature of gas stream from hypothetical isentropic compressor i (K)
T_{max} = maximum allowable operating temperature for all compressors (K)
T_n = outlet temperature of gas stream from compressor/cooler system (K)
w_i = normalized ideal compressor outlet temperature; $w_i \triangleq \frac{T''_i - T_0}{T_0}$
W_i = work done by compressor i (J/mol)
W^i = work done by a hypothetical isentropic compressor (J/mol)
Z = compressibility factor

Subscripts

id = ideal (isentropic) compressor
in = inlet stream to process unit
out = outlet stream to process unit
r = real compressor

Superscripts

i = ideal (isentropic) compressor case
o = initial state of fluid stream
R = reference state of fluid stream

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APPENDIX

Proof of Lemma.

a. The changes in molar enthalpy and molar entropy of a real fluid from the state (T^{o}, P^{o}) to the state (T, P) are derived from the exact differentials of molar enthalpy and molar entropy in (T, P) space[12]

$$\left\{ \begin{aligned} dH &= C_{p}(T,P)dT + V(T,P)(1-\beta(T,P)T)dP \\ dS &= \frac{C_{p}(T,P)}{T}dT - \beta(T,P)V(T,P)dP \end{aligned} \right\}$$

where $\beta(T,P) \triangleq \frac{1}{V(T,P)}\frac{\partial V(T,P)}{\partial T}$, $\kappa(T,P) \triangleq -\frac{1}{V(T,P)}\frac{\partial V(T,P)}{\partial P}$.
Consider the reference state (T^{R}, P^{R}) where $P^{R} \to 0$. Since H, S are state functions, it then holds

$$H(T,P)-H(T^{o},P^{o}) = \left[\begin{array}{c} [H(T,P)-H(T,P^{R})]+[H(T,P^{R})-H(T^{R},P^{R})]+ \\ [H(T^{R},P^{R})-H(T^{o},P^{R})]+[H(T^{o},P^{R})-H(T^{o},P^{o})] \end{array} \right] \Rightarrow$$

$$H(T,P)-H(T^{o},P^{o}) = \left[\begin{array}{c} \int_{P^{R}}^{P} V\left(T,P'\right)\left(1-\beta\left(T,P'\right)T\right)dP' + \int_{T^{R}}^{T} C_{p}\left(T',P^{R}\right)dT' + \\ + \int_{T^{o}}^{T^{R}} C_{p}\left(T',P^{R}\right)dT' + \int_{P^{o}}^{P^{R}} V\left(T^{o},P'\right)\left(1-\beta\left(T^{o},P'\right)T^{o}\right)dP' \end{array} \right] \Rightarrow$$

$$H(T,P)-H(T^{o},P^{o}) = \left[\begin{array}{c} \int_{P^{R}}^{P} V\left(T,P'\right)\left(1-\beta\left(T,P'\right)T\right)dP' + \int_{T^{o}}^{T} C_{p}\left(T',P^{R}\right)dT' + \\ + \int_{P^{o}}^{P^{R}} V\left(T^{o},P'\right)\left(1-\beta\left(T^{o},P'\right)T^{o}\right)dP' \end{array} \right]$$

$$S(T,P)-S(T^{o},P^{o})=\begin{bmatrix}[S(T,P)-S(T,P^{R})]+[S(T,P^{R})-S(T^{R},P^{R})]+\\ [S(T^{R},P^{R})-S(T^{o},P^{R})]+[S(T^{o},P^{R})-S(T^{o},P^{o})]\end{bmatrix}\Rightarrow$$

$$S(T,P)-S(T^{o},P^{o})=\begin{bmatrix}-\int_{P^{R}}^{P}\beta\left(T,P'\right)V\left(T,P'\right)dP'+\int_{T^{R}}^{T}\frac{C_{p}\left(T',P^{R}\right)}{T'}dT'+\\ +\int_{T^{o}}^{T^{R}}\frac{C_{p}\left(T',P^{R}\right)}{T'}dT'-\int_{P^{o}}^{P^{R}}\beta\left(T^{o},P'\right)V\left(T^{o},P'\right)dP'\end{bmatrix}\Rightarrow$$

$$S(T,P)-S(T^{o},P^{o})=\begin{bmatrix}-\int_{P^{R}}^{P}\beta\left(T,P'\right)V\left(T,P'\right)dP'+\int_{T^{o}}^{T}\frac{C_{p}\left(T',P^{R}\right)}{T'}dT'+\\ -\int_{P^{o}}^{P^{R}}\beta\left(T^{o},P'\right)V\left(T^{o},P'\right)dP'\end{bmatrix}$$

b. Equation 3 holds by assumption of constant compressibility factor. The proof for Eqs. 4–6 is straightforward. To establish Eq. 7, we proceed as follows:
The constant-pressure and constant-volume heat capacities are related as follows

$$C_{p}(T,P)-C_{v}(T,P)-\frac{\beta(T,P)^{2}}{\kappa(T,P)}TV=0$$

For a gas with a constant compressibility factor Z, by Eq. 4 it holds:
$\beta(T,P)=\beta(T)=\frac{1}{T}$, $\kappa(T,P)=\kappa(P)=\frac{1}{P}$. Then, the above relation becomes

$$C_{p}(T,P)-C_{v}(T,P)-\frac{PV}{T}=0\overset{Z=\frac{PV}{RT}}{\Longleftrightarrow}C_{p}(T,P)-C_{v}(T,P)=RZ.$$

c. The inlet and outlet molar entropies of a reversible adiabatic (ideal) compressor of a gas featuring a constant compressibility factor are equal to one another. Thus, considering the ideal compressor's inlet and outlet temperatures and pressures to be T_{in},T'_{out} and P_{in},P_{out}, respectively, Eq. 6 yields the following

$$S\left(T'_{out},P_{out}\right)=S(T_{in},P_{in})\Longleftrightarrow\int_{T_{in}}^{T'_{out}}\frac{C_{p}\left(T'\right)}{RT'}dT'=Z\ln\left(\frac{P_{out}}{P_{in}}\right)$$

An energy balance for the ideal compressor, combined with its adiabatic nature yields:
$W_{id}=H\left(T'_{out},P_{out}\right)-H(T_{in},P_{in})$. By Eq. 5, it then holds $W_{id}=R\int_{T_{in}}^{T'_{out}}\frac{C_{p}\left(T'\right)}{R}dT'$.
d. An energy balance for the real compressor, combined with its adiabatic nature yields:
$W_{r}=H(T_{out},P_{out})-H(T_{in},P_{in})$. By Eq. 5, it then holds $W_{r}=R\int_{T_{in}}^{T_{out}}\frac{C_{p}\left(T'\right)}{R}dT'$.
By the definition of compressor efficiency, it holds: $\eta=\frac{W_{id}}{W_{r}}$.
Then, the work equations developed in part (c) and in the earlier part of (d) yield:

$$\eta=\frac{H\left(T'_{out},P_{out}\right)-H(T_{in},P_{in})}{H(T_{out},P_{out})-H(T_{in},P_{in})}=\frac{\int_{T_{in}}^{T'_{out}}C_{p}\left(T'\right)dT'}{\int_{T_{in}}^{T_{out}}C_{p}\left(T'\right)dT'}$$

e. Straightforward.
f. Consider a constant compressibility factor gas with a temperature-independent (constant), constant-pressure, ideal gas, heat capacity $C_{p}(T,P^{R})\triangleq C_{p}(T)=C_{p}=$constant. Let this gas be compressed through an adiabatic ideal compressor, and through an adiabatic real compressor with known efficiency $\eta\in(0,1)$. Let the inlet temperatures, and inlet and outlet pressures T_{in},P_{in},P_{out} to both compressors be the same, and let the outlet temperatures be denoted as T'_{out},T_{out}, respectively. Combining Eqs. 7 and 8, and the fact that $C_{p}=$constant yields

$$\left\{\begin{array}{l}W_{id}=\int_{T_{in}}^{T'_{out}}C_{p}dT'\\ ZR\ln\left(\frac{P_{out}}{P_{in}}\right)=\int_{T_{in}}^{T'_{out}}\frac{C_{p}}{T'}dT'\end{array}\right\}\overset{C_{p}=\text{constant}}{\Longleftrightarrow}$$

$$\left\{\begin{array}{l}W_{id}=C_{p}\left(T'_{out}-T_{in}\right)\\ ZR\ln\left(\frac{P_{out}}{P_{in}}\right)=C_{p}\ln\left(\frac{T'_{out}}{T_{in}}\right)\end{array}\right\}\underset{k\triangleq\frac{C_{p}}{C_{v}}}{\overset{C_{p}-C_{v}=RZ}{\Longleftrightarrow}}$$

$$\left\{\begin{array}{l}W_{id}=C_{p}\left(T'_{out}-T_{in}\right)\\ \frac{k-1}{k}\ln\left(\frac{P_{out}}{P_{in}}\right)=\ln\left(\frac{T'_{out}}{T_{in}}\right)\end{array}\right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{l} W_{\mathrm{id}}=ZR\frac{k}{k-1}T_{\mathrm{in}}\left(\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)^{\frac{k-1}{k}}-1\right) \\ T'_{\mathrm{out}}=T_{\mathrm{in}}\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)^{\frac{k-1}{k}} \end{array} \right\}$$

The work of the real compressor and its outlet temperature then become

$$W_{\mathrm{r}}=\frac{W_{\mathrm{id}}}{\eta}=\frac{1}{\eta}ZR\frac{k}{k-1}T_{\mathrm{in}}\left(\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)^{\frac{k-1}{k}}-1\right)$$

$$\begin{aligned} \eta&=\frac{W_{\mathrm{id}}}{W_{\mathrm{r}}}=\frac{C_{\mathrm{p}}\left(T'_{\mathrm{out}}-T_{\mathrm{in}}\right)}{C_{\mathrm{p}}(T_{\mathrm{out}}-T_{\mathrm{in}})} \\ &=\frac{\left(T'_{\mathrm{out}}-T_{\mathrm{in}}\right)}{(T_{\mathrm{out}}-T_{\mathrm{in}})}\Rightarrow T_{\mathrm{out}} \\ &=T_{\mathrm{in}}+\frac{\left(T'_{\mathrm{out}}-T_{\mathrm{in}}\right)}{\eta} \\ &=T_{\mathrm{in}}\left(1+\frac{\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)^{\frac{k-1}{k}}-1}{\eta}\right)\text{O.E.}\Delta. \end{aligned}$$

Proof of Theorem 1.

1. The compression ratio $\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}>1$ is known. Let $C_{\mathrm{p}}:\Re^+\rightarrow\Re^+, C_{\mathrm{p}}:T\rightarrow C_{\mathrm{p}}(T)\triangleq C_{\mathrm{p}}(T,P^o)\geq 0\,\forall T\in\Re^+$. Then, Eq. 8 implies $\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out}}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=ZR\ln\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)>0$. Since $C_{\mathrm{p}}(T)>0\,\forall T\in\left[T_{\mathrm{in}},T'_{\mathrm{out}}\right]\wedge T_{\mathrm{in}}>0$, it then holds $\frac{C_{\mathrm{p}}(T)}{T}>0\,\forall T\in\left[T_{\mathrm{in}},T'_{\mathrm{out}}\right]$. Then, since $Z>0$ and $\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}>1$, it holds $0<ZR\ln\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)=\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out}}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'\Rightarrow T'_{\mathrm{out}}>T_{\mathrm{in}}$. Consider now, for any arbitrary but fixed $T_{\mathrm{in}}>0$, that $\exists\left(T'_{\mathrm{out},a},T'_{\mathrm{out},b}\right)$ $:T'_{\mathrm{out},a}>T'_{\mathrm{out},b}>T_{\mathrm{in}}\wedge ZR\ln\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)=\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out},a}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out},b}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'$. Then, $\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out},b}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'-\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out},a}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=0\Rightarrow\int_{T'_{\mathrm{out},a}}^{T'_{\mathrm{out},b}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=0\overset{\frac{C_{\mathrm{p}}\left(T'\right)}{T'}>0\forall T'>0}{\Longleftrightarrow}T'_{\mathrm{out},a}=T'_{\mathrm{out},b}$. This is a contradiction. Therefore, each T_{in} maps to a unique corresponding T'_{out}. In turn, this implies that there exists a function $f:\Re^+\rightarrow\Re^+, f:T_{\mathrm{in}}\rightarrow T'_{\mathrm{out}}=f(T_{\mathrm{in}})$.

It was established above that $\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out}}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=\int_{T_{\mathrm{in}}}^{f(T_{\mathrm{in}})}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=ZR\ln\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)>0\,\forall T_{\mathrm{in}}>0$ establishes the existence of a function $f:\Re^+\rightarrow\Re^+, f:T_{\mathrm{in}}\rightarrow T'_{\mathrm{out}}=f(T_{\mathrm{in}})$. Then

$$\begin{aligned} &\frac{d\left(\int_{T_{\mathrm{in}}}^{f(T_{\mathrm{in}})}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'\right)}{dT_{\mathrm{in}}}=\frac{d\left(ZR\ln\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)\right)}{dT_{\mathrm{in}}} \\ &=0\,\forall T_{\mathrm{in}}>0\Rightarrow\int_{T_{\mathrm{in}}}^{f(T_{\mathrm{in}})}\frac{\partial\left(\frac{C_{\mathrm{p}}\left(T'\right)}{T'}\right)}{\partial T_{\mathrm{in}}}dT' \\ &+\frac{C_{\mathrm{p}}(f(T_{\mathrm{in}}))}{f(T_{\mathrm{in}})}\frac{df(T_{\mathrm{in}})}{dT_{\mathrm{in}}}-\frac{C_{\mathrm{p}}(T_{\mathrm{in}})}{T_{\mathrm{in}}}\frac{dT_{\mathrm{in}}}{dT_{\mathrm{in}}} \\ &=0\,\forall T_{\mathrm{in}}>0\Rightarrow \\ &\frac{df(T_{\mathrm{in}})}{dT_{\mathrm{in}}}=\frac{C_{\mathrm{p}}(T_{\mathrm{in}})}{T_{\mathrm{in}}}\frac{f(T_{\mathrm{in}})}{C_{\mathrm{p}}(f(T_{\mathrm{in}}))} \\ &=\frac{C_{\mathrm{p}}(T_{\mathrm{in}})}{T_{\mathrm{in}}}\frac{T'_{\mathrm{out}}}{C_{\mathrm{p}}\left(T'_{\mathrm{out}}\right)}>0\,\forall T_{\mathrm{in}}>0.\text{O.E.}\Delta. \end{aligned}$$

2. In part (1) of Theorem 1, it was shown that the relation $\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out}}}\frac{C_{\mathrm{p}}\left(T'\right)}{T'}dT'=ZR\ln\left(\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}}\right)>0$ establishes the existence of a function $f:\Re^+\rightarrow\Re^+, f:T_{\mathrm{in}}\rightarrow T'_{\mathrm{out}}=f(T_{\mathrm{in}})$ with derivative $\frac{df(T_{\mathrm{in}})}{dT_{\mathrm{in}}}=\frac{C_{\mathrm{p}}(T_{\mathrm{in}})}{T_{\mathrm{in}}}\frac{f(T_{\mathrm{in}})}{C_{\mathrm{p}}(f(T_{\mathrm{in}}))}=\frac{C_{\mathrm{p}}(T_{\mathrm{in}})}{T_{\mathrm{in}}}\frac{T'_{\mathrm{out}}}{C_{\mathrm{p}}\left(T'_{\mathrm{out}}\right)}>0\,\forall T_{\mathrm{in}}>0$. Then, the function $\Delta H:\Re^+\rightarrow\Re^+, \Delta H:T_{\mathrm{in}}\rightarrow\Delta H(T_{\mathrm{in}})\triangleq\int_{T_{\mathrm{in}}}^{T'_{\mathrm{out}}}C_{\mathrm{p}}\left(T'\right)\,dT'=\int_{T_{\mathrm{in}}}^{f(T_{\mathrm{in}})}C_{\mathrm{p}}\left(T'\right)dT'$ is differentiable $\forall T_{\mathrm{in}}>0$, with derivative

$$\begin{aligned} \frac{d(\Delta H(T_{\mathrm{in}}))}{dT_{\mathrm{in}}}&=\frac{d\left(\int_{T_{\mathrm{in}}}^{f(T_{\mathrm{in}})}C_{\mathrm{p}}\left(T'\right)dT'\right)}{dT_{\mathrm{in}}}=\int_{T_{\mathrm{in}}}^{f(T_{\mathrm{in}})}\frac{\partial\left(C_{\mathrm{p}}\left(T'\right)\right)}{\partial T_{\mathrm{in}}}dT' \\ &+C_{\mathrm{p}}(f(T_{\mathrm{in}}))\frac{df(T_{\mathrm{in}})}{dT_{\mathrm{in}}}-C_{\mathrm{p}}(T_{\mathrm{in}})\frac{dT_{\mathrm{in}}}{dT_{\mathrm{in}}} \\ &=C_{\mathrm{p}}(f(T_{\mathrm{in}}))\frac{df(T_{\mathrm{in}})}{dT_{\mathrm{in}}}-C_{\mathrm{p}}(T_{\mathrm{in}}) \\ &=C_{\mathrm{p}}(f(T_{\mathrm{in}}))\frac{C_{\mathrm{p}}(T_{\mathrm{in}})}{T_{\mathrm{in}}}\frac{f(T_{\mathrm{in}})}{C_{\mathrm{p}}(f(T_{\mathrm{in}}))} \\ &-C_{\mathrm{p}}(T_{\mathrm{in}})=C_{\mathrm{p}}(T_{\mathrm{in}})\left(\frac{f(T_{\mathrm{in}})}{T_{\mathrm{in}}}-1\right) \\ &=C_{\mathrm{p}}(T_{\mathrm{in}})\left(\frac{T'_{\mathrm{out}}}{T_{\mathrm{in}}}-1\right)\forall T_{\mathrm{in}}>0 \end{aligned} \tag{24}$$

Based on the above, $\frac{d(\Delta H(T_{\mathrm{in}}))}{dT_{\mathrm{in}}}>0\Longleftrightarrow C_{\mathrm{p}}(T_{\mathrm{in}})\left(\frac{T'_{\mathrm{out}}}{T_{\mathrm{in}}}-1\right)>0\overset{C_{\mathrm{p}}(T_{\mathrm{in}})>0\forall T_{\mathrm{in}}>0}{\Longleftrightarrow}\frac{T'_{\mathrm{out}}}{T_{\mathrm{in}}}-1>0\Longleftrightarrow T'_{\mathrm{out}}>T_{\mathrm{in}}$, which is true by the proof of (1). Thus ΔH is a monotonically increasing function of T_{in}. O.E.Δ.

Proof of Theorem 2. The sequential nature of a compressor/cooler sequence allows an embedded representation of the considered optimization problem. In the interior optimization problem, all temperatures can be considered fixed, but unknown, at arbitrary (feasible) values, except for T'_{k-1}, T''_k, T_k, which are the interior optimization problem's variables. Then problem (19) can be rewritten as

$$
v=\min_{\substack{\{T_i,T_i''\}^n_{i=1,\, i\neq k} \\ \{T_i'\}^n_{i=1,\, i\neq k-1}}}
\left[
\begin{array}{l}
\sum_{\substack{i=1\\ i\neq k}}^{n}\left[FC^{\text{cap.}}_{\text{compr.}}\left(\frac{\dot{n}\cdot R}{\eta_i}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'\right)^{a}+C^{\text{oper.}}_{\text{compr.}}\left(\frac{\dot{n}\cdot R}{\eta_i}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'\right)\right] \\
+\sum_{\substack{i=1\\ i\neq k,k-1}}^{n}C^{\text{oper.}}_{\text{cooler}}\left(\frac{\dot{n}\cdot R\int_{T'_i}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{\text{c,out}}-T_{\text{c,in}}\right)}\right) \\
+\min_{T_k,T'_{k-1},T''_k}\left[
\begin{array}{l}
FC^{\text{cap.}}_{\text{compr.}}\left(\frac{\dot{n}\cdot R}{\eta_k}\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{R}dT'\right)^{a}+C^{\text{oper.}}_{\text{compr.}}\left(\frac{\dot{n}\cdot R}{\eta_k}\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{R}dT'\right) \\
+C^{\text{oper.}}_{\text{cooler}}\left(\frac{\dot{n}\cdot R\int_{T'_k}^{T_k}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{\text{c,out}}-T_{\text{c,in}}\right)}\right)+C^{\text{oper.}}_{\text{cooler}}\left(\frac{\dot{n}\cdot R\int_{T'_{k-1}}^{T_{k-1}}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{\text{c,out}}-T_{\text{c,in}}\right)}\right)
\end{array}
\right]
\end{array}
\right] \tag{A1}
$$

s.t.

$$Z\ln\left(\frac{P_n}{P_0}\right)=\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{RT'}dT'+\sum_{i\neq k}^{i=1\ n}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{RT'}dT'$$

$$0<\eta_k=\frac{\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{R}dT'}{\int_{T'_{k-1}}^{T_k}\frac{C_{\text{p}}(T')}{R}dT'}<1,\ 0<\eta_i=\frac{\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{R}dT'}{\int_{T'_{i-1}}^{T_i}\frac{C_{\text{p}}(T')}{R}dT'}<1,\ i=1,n;\ i\neq k$$

$$T'_{k-1}\leq T''_k\leq T_k\leq T_{\max}$$

$$T_0\leq T'_k\leq T_k,\ T_0\leq T'_{k-1}\leq T_{k-1}$$

$$T'_{i-1}\leq T''_i\leq T_i\leq T_{\max}<\infty,\ i=1,n;\ i\neq k,\ T'_0=T_0$$

$$0<T_0\leq T'_i\leq T_i,\ i=1,n-1;\ i\neq k,k-1,\ T'_n=T_0$$

The inner level of the above embedded optimization problem can then be stated as follows

$$
\min_{T_k,T'_{k-1},T''_k}\left[
\begin{array}{l}
FC^{\text{cap.}}_{\text{compr.}}\left(\frac{\dot{n}\cdot R}{\eta_k}\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{R}dT'\right)^{a}+C^{\text{oper.}}_{\text{compr.}}\left(\frac{\dot{n}\cdot R}{\eta_k}\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{R}dT'\right) \\
+C^{\text{oper.}}_{\text{cooler}}\left(\frac{\dot{n}\cdot R\int_{T'_k}^{T_k}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{\text{c,out}}-T_{\text{c,in}}\right)}\right)+C^{\text{oper.}}_{\text{cooler}}\left(\frac{\dot{n}\cdot R\int_{T'_{k-1}}^{T_{k-1}}\frac{C_{\text{p}}(T')}{R}dT'}{C_{\text{p,c}}\left(T_{\text{c,out}}-T_{\text{c,in}}\right)}\right)
\end{array}
\right]
$$

s.t.

$$0<\left[Z\ln\left(\frac{P_n}{P_0}\right)-\sum_{\substack{i=1\\ i\neq k}}^{n}\int_{T'_{i-1}}^{T''_i}\frac{C_{\text{p}}(T')}{RT'}dT'\right]=\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{RT'}dT' \tag{A2}$$

$$0<\eta_k=\frac{\int_{T'_{k-1}}^{T''_k}\frac{C_{\text{p}}(T')}{R}dT'}{\int_{T'_{k-1}}^{T_k}\frac{C_{\text{p}}(T')}{R}dT'}\leq 1$$

$$T'_{k-1}\leq T''_k\leq T_k\leq T_{\max}$$

$$T_0\leq T'_k\leq T_k,\ T_0\leq T'_{k-1}\leq T_{k-1}$$

However

$$\int_{T'_{k-1}}^{T_k} \frac{C_p(T')}{R} dT' = \frac{1}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' \Rightarrow \int_{T'_k}^{T_k} \frac{C_p(T')}{R} dT' = \int_{T'_{k-1}}^{T_k} \frac{C_p(T')}{R} dT' + \int_{T'_k}^{T'_{k-1}} \frac{C_p(T')}{R} dT' = \frac{1}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' + \int_{T'_k}^{T'_{k-1}} \frac{C_p(T')}{R} dT'.$$

The objective function of problem (A1) then becomes

$$FC^{\text{cap.}}_{\text{compr.}} \left(\frac{\dot{n} \cdot R}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' \right)^a + C^{\text{oper.}}_{\text{compr.}} \left(\frac{\dot{n} \cdot R}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' \right) +$$

$$+ C^{\text{oper.}}_{\text{cooler}} \left(\frac{\dot{n} \cdot R \left(\frac{1}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' + \int_{T'_k}^{T'_{k-1}} \frac{C_p(T')}{R} dT' \right)}{C_{p,c}(T_{c,\text{out}} - T_{c,\text{in}})} \right) + C^{\text{oper.}}_{\text{cooler}} \left(\frac{\dot{n} \cdot R \int_{T'_{k-1}}^{T_{k-1}} \frac{C_p(T')}{R} dT'}{C_{p,c}(T_{c,\text{out}} - T_{c,\text{in}})} \right)$$

$$= FC^{\text{cap.}}_{\text{compr.}} \left(\frac{\dot{n} \cdot R}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' \right)^a + \left(C^{\text{oper.}}_{\text{compr.}} + \frac{C^{\text{oper.}}_{\text{cooler}}}{C_{p,c}(T_{c,\text{out}} - T_{c,\text{in}})} \right) \frac{\dot{n} \cdot R}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' + C^{\text{oper.}}_{\text{cooler}} \frac{\dot{n} \cdot R \int_{T'_k}^{T_{k-1}} \frac{C_p(T')}{R} dT'}{C_{p,c}(T_{c,\text{out}} - T_{c,\text{in}})}$$

Since T'_k, T_{k-1} are considered fixed but unknown, the optimization problem (A1) can be rewritten as

$$C^{\text{oper.}}_{\text{cooler}} \frac{\dot{n} \cdot R \int_{T'_k}^{T_{k-1}} \frac{C_p(T')}{R} dT'}{C_{p,c}(T_{c,\text{out}} - T_{c,\text{in}})} + \min_{T_k, T'_{k-1}, T''_k} FC^{\text{cap.}}_{\text{compr.}} \left(\frac{\dot{n} \cdot R}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT' \right)^a + \left(C^{\text{oper.}}_{\text{compr.}} + \frac{C^{\text{oper.}}_{\text{cooler}}}{C_{p,c}(T_{c,\text{out}} - T_{c,\text{in}})} \right) \frac{\dot{n} \cdot R}{\eta_k} \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT'$$

s.t.

$$0 < \left[Z \ln\left(\frac{P_n}{P_0}\right) - \sum_{\substack{i=1 \\ i \neq k}}^{n} \int_{T'_{i-1}}^{T''_i} \frac{C_p(T')}{RT'} dT' \right] = \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{RT'} dT'$$

$$0 < \eta_k = \frac{\int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT'}{\int_{T'_{k-1}}^{T_k} \frac{C_p(T')}{R} dT'} \leq 1$$

$$T'_{k-1} \leq T''_k \leq T_k \leq T_{\max}$$

$$T_0 \leq T'_k \leq T_k, T_0 \leq T'_{k-1} \leq T_{k-1}$$

The constraint $0 < [Z \ln\left(\frac{P_n}{P_0}\right) - \sum_{\substack{i=1 \\ i \neq k}}^{n} \int_{T'_{i-1}}^{T''_i} \frac{C_p(T')}{RT'} dT'] = \int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{RT'} dT'$ and part (2) of Theorem 1 suggest that $\int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT'$ is a monotonically increasing function of T'_{k-1}. Since the objective function is a positive-weighted combination of $\int_{T'_{k-1}}^{T''_k} \frac{C_p(T')}{R} dT'$ and its ath power, its global optimum occurs at the minimum possible value of T'_{k-1}. Examination of the problem's constraints suggests that this value is T_0. Thus, at the global optimum it holds $T'_{k-1} = T_0$. Repeating this process for all compressors establishes the theorem's claim. O.E.Δ.

Proof of Theorem 3. ($\Leftarrow$)Let $C \leq \prod_{i=1}^{n} (\eta_i D + 1)$. Define $w_i \triangleq \phi \eta_i D, \forall i = 1, n$, where $\phi \in [0, 1]$ is such that $C = \prod_{i=1}^{n} (\phi \eta_i D + 1)$. Such a $\phi \in [0, 1]$ exists, since the function $g : [0, 1] \to \Re, g : \phi \to g(\phi) \triangleq C - \prod_{i=1}^{n} (\phi \eta_i D + 1)$ is continuous on $[0, 1]$, and has values $g(0) \triangleq C - 1 > 0 \wedge g(1) \triangleq C - \prod_{i=1}^{n} (\eta_i D + 1) \leq 0$. Then, the variable vector $\{w_i\}_1^n \triangleq \{\phi \eta_i D\}_1^n$, where $\phi \in [0, 1]$ is such that $C = \prod_{i=1}^{n} (\phi \eta_i D + 1)$, is a feasible point for v.
($\Rightarrow$) Let v be feasible. For any feasible variable vector $\{w_i\}_1^n$, it holds $\{\prod_{i=1}^{n} (w_i + 1) - C = 0 \wedge 0 \leq w_i \leq \eta_i D,\ i = 1, n\} \Rightarrow 1 \leq \prod_{i=1}^{n} (w_i + 1) = C \leq \prod_{i=1}^{n} (\eta_i D + 1)$. O.E.Δ.

Proof of Theorem 4. It is easy to establish that the above optimization problem's feasible region (which is nonempty) is closed, and bounded, that its objective function and constraint defining functions are all differentiable throughout the feasible region, and that all its feasible points are regular. Then the problem's optimum exists and the following first-order necessary optimality conditions are defined based on the Lagrangian $L(w,\lambda,\mu,v) \triangleq \sum_{i=1}^{n}\frac{1}{\eta_i}\cdot w_i+\lambda\cdot\left(\prod_{i=1}^{n}(w_i+1)-C\right)+\sum_{i=1}^{n}\mu_i\cdot(-w_i)+\sum_{i=1}^{n}v_i\cdot(w_i-\eta_i D)$. They are

$$\left\{\begin{array}{l}\dfrac{\partial L(w,\lambda,\mu,v)}{\partial w_k}=\dfrac{1}{\eta_k}+\lambda\cdot\prod_{\substack{i=1\\i\neq k}}^{n}(w_i+1)-\mu_k+v_k=0\,\forall k=1,n\\ \prod_{i=1}^{n}(w_i+1)-C=0,\,0\leq w_i\leq\eta_i D\,\forall i=1,n\\ \mu_i\geq 0\,\forall i=1,n,\,v_i\geq 0\,\forall i=1,n,\,\mu_i w_i=0\,\forall i=1,n,\,v_i(w_i-\eta_i D)=0\,\forall i=1,n,\end{array}\right\}\overset{w_k+1\geq 1\forall k=1,n}{\Longleftrightarrow}$$

$$\left\{\begin{array}{l}\dfrac{1}{\eta_k}(w_k+1)+\lambda\cdot C=0\,\forall k\in S_I^w\\ \left(\dfrac{1}{\eta_i}-\mu_i\right)+\lambda\cdot C=0\,\forall i\in S_0^w\\ \left(\dfrac{1}{\eta_j}+v_j\right)(\eta_j D+1)+\lambda\cdot C=0\,\forall j\in S_D^w\\ \prod_{l\in S_I^w}[-\lambda\eta_l C]\prod_{m\in S_D^w}(\eta_m D+1)-C=0\\ 0<w_k<\eta_k D\,\forall k\in S_I^w\\ \mu_i\geq 0\,\forall i\in S_0^w,\,v_j\geq 0\,\forall j\in S_D^w\end{array}\right\}\Longleftrightarrow$$

$$\left\{\begin{array}{l}0<w_k=\eta_k\left[\dfrac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\right]^{\frac{1}{N_I^w}}-1<\eta_k D\,\forall k\in S_I^w\\ \mu_i=\dfrac{1}{\eta_i}-\left[\dfrac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\right]^{\frac{1}{N_I^w}}\geq 0\,\forall i\in S_0^w\\ v_j=\left[\dfrac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\right]^{\frac{1}{N_I^w}}\dfrac{1}{(\eta_j D+1)}-\dfrac{1}{\eta_j}\geq 0\,\forall j\in S_D^w\\ \lambda=\dfrac{-1}{C}\left[\dfrac{C}{\left(\prod_{l\in S_I^w}\eta_l\right)\left(\prod_{m\in S_D^w}(\eta_m D+1)\right)}\right]^{\frac{1}{N_I^w}}<0\end{array}\right.$$

The associated objective function value is

$$v = A \cdot \left[N_I^w \left[\frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right)\left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \right]^{\frac{1}{N_I^w}} - \sum_{k \in S_I^w} \frac{1}{\eta_k} + N_D^w \cdot D \right]$$

The above necessary optimality conditions then imply

$$\left\{ \begin{array}{l} w_k = \left[\eta_k \left[\frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right)\left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \right]^{\frac{1}{N_I^w}} - 1 \right] \in (0, \eta_k D) \forall k \in S_I^w \\ \wedge \\ w_i = 0 \, \forall i \in S_0^w \\ w_j = \eta_j D \, \forall j \in S_D^w \\ \left\{ \begin{array}{l} \left(\frac{1}{\eta_k}\right)^{N_I^w} < \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} < \left(D + \frac{1}{\eta_k}\right)^{N_I^w} \forall k \in S_I^w \\ \left(\frac{1}{\eta_i}\right)^{N_I^w} \geq \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \forall i \in S_0^w \\ \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \geq \left(D + \frac{1}{\eta_j}\right)^{N_I^w} \forall j \in S_D^w \end{array} \right\} \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{l} \max_{k \in S_I^w} \left(\frac{1}{\eta_k}\right)^{N_I^w} < \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} < \min_{k \in S_I^w} \left(D + \frac{1}{\eta_k}\right)^{N_I^w} \\ \min_{i \in S_0^w} \left(\frac{1}{\eta_i}\right)^{N_I^w} \geq \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \\ \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \geq \max_{j \in S_D^w} \left(D + \frac{1}{\eta_j}\right)^{N_I^w} \end{array} \right\} \Rightarrow$$

$$\max\left(\left(D + \frac{1}{\min_{j \in S_D^w} \eta_j}\right)^{N_I^w}, \left(\frac{1}{\min_{k \in S_I^w} \eta_k}\right)^{N_I^w} \right) \leq \frac{C}{\left(\prod_{l \in S_I^w} \eta_l\right) \cdot \left(\prod_{m \in S_D^w} (\eta_m D + 1)\right)} \leq$$

$$\leq \min\left(\left(\frac{1}{\max_{i \in S_0^w} \eta_i}\right)^{N_I^w}, \left(D + \frac{1}{\max_{k \in S_I^w} \eta_k}\right)^{N_I^w} \right) \text{if} N_0^w \neq 0 \wedge N_D^w \neq 0$$

The remaining conditions are straightforward to establish.O.E.Δ.

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